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Basic circular functions

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SUMMARY

A basic analogue of the circular functions is introduced from the point of view of the orthogonality of these functions with respect to basic integration. This leads to the formation of a new basic analogue of the exponential function. The asymptotic distribution of the zeros of the basic circular functions is briefly discussed and associated numerical results tabulated.

1. INTRODUCTION

In a recent paper, the author has discussed a basic analogue of the Bessel-Clifford functions, where the ordinary number system has been replaced by the basic number system. See Exton (1). That is, the number a, for example, is replaced by the quantity $[a]$ given by

$$(1.1) \qquad [a]=(1-q^a)/(1-q),$$

where a is any number, and q is any number, real or complex, called the base. It will be seen that if n is a positive integer,

$$(1.2) \qquad [n]=1+q+q^2+\ldots+q^{n-1}.$$

See also Slater (4), Chapter 3.

Jackson's basic analogues of differentiation and integration have also been employed.

These operations are denoted by

$$(1.3) \qquad \hat{B}_{q,x}\phi(x)=\frac{\phi(x)-\phi(qx)}{x(1-q)}$$

and

(1.4) $\quad \mathrm{S}\,\phi(x)d(qx),$

where S is the inverse of $\hat{B}$. See Jackson (3).

All operations with basic numbers reduce to their well-known counterparts in ordinary analysis when the limit is taken as $q \to 1$. When there is no possibility of confusion, the subscripts q and x associated with the symbol for basic differentiation will be omitted. The following basic analogue of the simple Sturm-Liouville system of the second order was introduced in Exton (1).

LEMMA: Suppose that the real functions $r(x)$, $l(x)$ and $w(x)$ possess the appropriate number of q-derivatives and that q is real and positive. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions corresponding to distinct eigenvalues λ_m, λ_n of the boundary-value system

$$(1.5) \qquad \begin{cases} \hat{B}\{r(x)\hat{B}y(x)\} + \{l(x) + \lambda w(x)\}y(qx) = 0 \\ h_1 y + h_2 \hat{B} y = 0 \text{ at } x = g \\ k_1 y + k_2 \hat{B} y = 0 \text{ at } x = h, \end{cases}$$

h_1, h_2, k_1 and k_2 being constants. Then $y_m(qx)$ and $y_n(qx)$ are q-orthogonal on the closed interval $g \leq x \leq h$ with respect to the weight function $w(x)$. That is

$$(1.6) \qquad \underset{g}{\overset{h}{\mathrm{S}}}\, w(x) y_m(qx) y_n(qx) d(qx) = 0, \; m \neq n.$$

For a proof of the above lemma and further details of the basic analogues of differentiation and integration, the reader is referred to Exton (1).

The simplest case of (1.5) is furnished by the basic differential equation

$$(1.7) \qquad \hat{B}^2 y + \lambda^2 y(qx) = 0$$

together with the appropriate boundary conditions. When q tends to unity, the equation (1.7) reduces to the differential equation satisfied by the circular functions if the eigenvalue λ is real. The solutions of (1.7) will consist of the basic circular functions given below. Unless otherwise indicated, all quantities are taken to be real and the base $q > 0$ in the rest of this paper.

2. A BASIC EXPONENTIAL FUNCTION

Before proceeding directly to the basic circular functions, we examine a basic analogue of the exponential function. Consider the basic differential equation of the first order

$$(2.1) \qquad \hat{B}y = y(q^{\frac{1}{2}}x).$$

Let

$$(2.2) \qquad y = \sum_{r=0}^{\infty} a_r x^{r+\varrho}, \; a_0 \neq 0,$$

whence term-by-term basic differentiation gives

$$\hat{B}y = \sum_{r=0}^{\infty} a_r[r+\varrho]x^{r+\varrho-1}. \tag{2.3}$$

The indicial equation is

$$[\varrho]=0 \tag{2.4}$$

whose principal solution is

$$\varrho=0. \tag{2.5}$$

Hence, the coefficients $\{a_r\}$ are related by the recurrence formula

$$a_{r+1}=a_r q^{\frac{1}{2}r}/[r+1], \tag{2.6}$$

and so, if $a_0=1$,

$$a_r=q^{\frac{1}{2}r(r-1)}/[r]!. \tag{2.7}$$

The standard form of the solution of (2.1) is now

$$y=\sum_{r=0}^{\infty} q^{\frac{1}{2}r(r-1)}x^r/[r]!, \tag{2.8}$$

and this series is written $E(q;x)$ which is the particular basic analogue of the exponential function under consideration. It must be stressed here that an infinite number of basic analogues of the exponential function, or, indeed of any function are possible. It will be seen that $E(q;x)$ reduces to e^x when $q\to 1$. In addition, the series (2.8) converges for all finite values of $|x|$, as may be established by a simple ratio test. If we replace q by $1/q$, $E(q;x)$ is unchanged in form, so that

$$E(q;x)=E(1/q;x). \tag{2.9}$$

3. THE BASIC CIRCULAR FUNCTIONS

We define the functions Cos $(q;x)$ and Sin $(q;x)$ by means of the expression

$$E(q;ix)=\mathrm{Cos}\ (q;x)+i\ \mathrm{Sin}\ (q;x), \tag{3.1}$$

so that

$$\mathrm{Cos}\ (q;x)=\sum_{r=0}^{\infty}\frac{(-1)^r q^{r(r-\frac{1}{2})}x^{2r}}{[2r]!} \tag{3.2}$$

and

$$\mathrm{Sin}\ (q;x)=\sum_{r=0}^{\infty}\frac{(-1)^r q^{r(r+\frac{1}{2})}x^{2r+1}}{[2r+1]!}. \tag{3.3}$$

If the series representations (3.2) and (3.3) are q-differentiated term by term, we see they are independent solutions of the equation

$$\hat{B}^2y+q^{\frac{1}{2}}y(qx)=0. \tag{3.4}$$

Furthermore, it follows that

$$(3.5) \qquad \hat{B} \operatorname{Cos}(q;x) = -q^{\frac{1}{2}} \operatorname{Sin}(q;q^{\frac{1}{2}}x)$$

and

$$(3.6) \qquad \hat{B} \operatorname{Sin}(q;x) = \operatorname{Cos}(q;q^{\frac{1}{2}}x).$$

From (2.9), we see that the basic circular functions Cos $(q;x)$ and Sin $(q;x)$ are invariant with respect to inversion of the base q.

If x is replaced by λx in (3.4), we obtain the equation

$$(3.5) \qquad \hat{B}^2 y + \lambda^2 q^{\frac{1}{2}} y(qx) = 0$$

which is of the same form as (1.7). The solutions of (3.5) are Cos $(q;\lambda x)$ and Sin $(q;\lambda x)$. When $q \to 1$, these two functions reduce to Cos λx and Sin λx respectively.

4. THE q-ORTHOGONALITY OF THE FUNCTIONS Cos $(q;x)$ AND Sin $(q;x)$

As pointed out in Section 1, equation (3.5) furnishes the simplest case of (1.5), so that the basic sine and cosine are orthogonal with respect to basic integration. Hence,

$$(4.1) \qquad \underset{\alpha}{\overset{\beta}{\mathrm{S}}} \operatorname{Sin}(q;\lambda_m qx) \operatorname{Sin}(q;\lambda_n qx) d(qx) = 0, \; m \neq n,$$

where α and β are zeros of the functions in the integrand. In particular,

$$(4.2) \qquad \underset{0}{\overset{c}{\mathrm{S}}} \operatorname{Sin}(q;\lambda_m qx) \operatorname{Sin}(q;\lambda qx) d(qx) = 0, \; m \neq n$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of the equation

$$(4.3) \qquad \operatorname{Sin}(q;cx) = 0.$$

It is thus possible to obtain a formal expansion of an arbitrary function as a series of basic circular functions, such as

$$(4.4) \qquad f(x) = \sum_{i=1}^{\infty} A_i \operatorname{Sin}(q;\lambda_i qx),$$

and the orthogonality relation (4.2) gives the formula

$$(4.5) \qquad A_i = \frac{\underset{0}{\overset{c}{\mathrm{S}}} \operatorname{Sin}(q;\lambda_i qx) f(x) d(qx)}{\underset{0}{\overset{c}{\mathrm{S}}} \{\operatorname{Sin}(q;\lambda_i qx)\}^2 d(qx)}.$$

By considering the limit

$$(4.6) \qquad \frac{1}{\lambda_i^2 - \lambda_j^2} \{\operatorname{Sin}(q;\lambda_i x) \hat{B} \operatorname{Sin}(q;\lambda_j x) - \operatorname{Sin}(q;\lambda_j x) \hat{B} \operatorname{Sin}(q;\lambda_i x)\}_{x=c}$$

as $j \to i$, we find that the denominator of the right-hand member of (4.5) may be written

$$\frac{-\lambda_i}{2} \operatorname{Cos}\,(q;\lambda_i qc) \frac{d}{d\lambda_i} \operatorname{Sin}\,(q;\lambda_i c). \tag{4.7}$$

For a more detailed discussion of the expansion of an arbitrary function in a series of basic orthogonal functions, see Exton (1), Section 6.

5. THE BASIC CIRCULAR FUNCTIONS AS SPECIAL CASES OF THE BASIC BESSEL-CLIFFORD FUNCTIONS

It is well known that the circular functions of ordinary analysis may be represented as Bessel-Clifford functions of order $\frac{1}{2}$ and $\frac{3}{2}$ with quadratic argument. It might be expected, therefore, that a similar relation would exist between the corresponding basic analogues. This in fact proves to be the case, but the connection is rather more involved. We note that in making any change of independent variable in the basic differential and integral operators, a change in the base must generally be made also. In fact, any changes other than replacing the independent variable by means of a power of itself does not appear to preserve the form of the expressions in question.

Referring to the series representation of Cos $(q;x)$, we note that the quantity $[2r]!$ may be expressed as

$$[1]\,[3]\,[5]\,\ldots\,[2r-1]\,[2]\,[4]\,[6]\,\ldots\,[2r]. \tag{5.1}$$

If we take a new base, $p=q^2$, then

$$[2r]=(1-p^r)/(1-p^{\frac{1}{2}})=[r]\,[\tfrac{1}{2}] \tag{5.2}$$

with respect to the new base.

Similarly,

$$[2r-1]=[r-\tfrac{1}{2}]\,[\tfrac{1}{2}]. \tag{5.3}$$

Hence, if

$$[a]_n=[a]\,[a+1]\,\ldots\,[a+n-1], \tag{5.4}$$

$$[2r]!=[\tfrac{1}{2}]_r[r]!\{[\tfrac{1}{2}]\}^2 \tag{5.5}$$

and

$$\operatorname{Cos}\,(q;x)=C_{\frac{1}{2}}(q^2;-x^2/[\tfrac{1}{2}]^2) \tag{5.6}$$

and

$$\operatorname{Sin}\,(q;x)=xC_{3/2}(q^2;-x^2/[\tfrac{1}{2}]^2), \tag{5.7}$$

where $C_a(q;x)$ is a basic Bessel-Clifford function.

The q-differential equation satisfied by Cos $(q;x)$ and Sin $(q;x)$ transforms into the basic Bessel-Clifford equation of order one-half and base q^2, that is

$$q^3x^2\hat{B}^2_{q^2,x^2}y+[\tfrac{3}{2}]\hat{B}_{q^2,x^2}y+q^{\frac{1}{2}}[\tfrac{1}{2}]y(q^2x)=0. \tag{5.8}$$

6. THE ASYMPTOTIC DISTRIBUTION OF THE ZEROS OF Cos $(q;x)$ AND Sin $(q;x)$

The basic exponential function $E_{1/q}(x)$ defined by

$$(6.1) \qquad E_{1/q}(x) = \sum_{r=0}^{\infty} \frac{x^r q^{\frac{1}{2}r(r-1)}}{[r]!}, \; q<1,$$

possesses zeros given by

$$(6.2) \qquad \mu_n = q^{1-n}(q-1)^{-1}, \; n = 1, 2, 3, \ldots.$$

See Jackson (2), p. 194. For sufficiently large values or r,

$$(6.3) \qquad [r] \approx (1-q)^{-1}$$

so that $E_{1/q}(x)$ is dominated by the series

$$(6.4) \qquad \sum_{r=0}^{\infty} q^{\frac{1}{2}r(r-1)}\{x(1-q)\}^r.$$

Similarly, Cos $(q;x)$ and Sin $(q;x)$ are dominated by the series

$$(6.5) \qquad \sum_{r=0}^{\infty} (-1)^r q^{r(r-1)}\{xq^{\frac{1}{4}}(1-q)\}^{2r}$$

and

$$(6.6) \qquad \sum_{r=0}^{\infty} (-1)^r q^{r(r-1)}\{xq^{\frac{3}{4}}(1-q)\}^{2r}$$

respectively. The comparison of (6.5) and (6.6) with (6.4) suggests that the zeros of Cos $(q;x)$ are given by

$$(6.7) \qquad \mu_n = q^{\frac{3}{4}-n}(1-q)^{-1}$$

and those of Sin $(q;x)$ by

$$(6.8) \qquad \mu_n = q^{\frac{1}{4}-n}(1-q)^{-1}, \; n = 1, 2, 3, \ldots$$

for sufficiently large values of n in each case.

This conjecture is confirmed by numerical computation, but the nearer q is to unity, the higher is the value of the integer n which must be taken before the expressions (6.7) and (6.8) are at all accurate. A result of this type would be expected because of the fact that the basic circular functions under consideration in this paper are special cases of the basic Bessel-Clifford functions.

It is thus evident that the basic circular functions possess the interesting property that their rate of oscillation decreases as the independent variable is increased. If μ_n is the n^{th} positive zero of these functions, then for sufficiently large n

$$\mu_n - \mu_{n-1} = q(\mu_{n+1} - \mu_n) \; q<1$$

or

$$(6.9) \qquad q(\mu_n - \mu_{n-1}) = \mu_{n+1} - \mu_n \; q>1.$$

In addition, it is observed numerically that the amplitude of the oscillations increases as the independent variable is increased. These properties are reflected in the following short tables:

Table 1. Zeros of Cos $(q;x)$

$q=0.9$	$q=0.5$	$q=0.1$
1.570	1.559	1.874
4.696	4.700	19.759
7.779	9.513	197.587
10.787	19.027	1975.866

Table 2. Zeros of Sin $(q;x)$

$q=0.9$	$q=0.5$	$q=0.1$
3.137	3.083	6.245
6.245	6.721	62.482
9.294	13.454	624.824
12.251	26.909	6248.237

Table 3. Turning points and turning values of Cos $(q;x)$

$q=0.9$		$q=0.5$		$q=0.1$	
Turning point	Turning value	Turning point	Turning value	Turning point	Turning value
3.141	−1.007	3.387	− −1.496	14.017	−27.169
6.254	1.029	7.946	10.922	161.440	1.62×10^{5}
9.309	−1.071	16.655	−478.173	1711.6	-1.15×10^{11}
12.276	1.143	34.223	95089.2	17675	8.92×10^{18}

Table 4. Turning points and turning values of Sin $(q;x)$

$q=0.9$		$q=0.5$		$q=0.1$	
Turning point	Turning value	Turning point	Turning value	Turning point	Turning value
1.572	1.002	1.668	1.085	3.597	2.396
4.703	−1.016	5.362	−3.146	48.468	−1136.471
7.791	1.047	11.549	59.252	528.285	7.54×10^{7}
10.806	−1.102	23.910	−5617.153	5511.341	-5.65×10^{14}

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